Semiclassical circular strings in AdS_5
and "long" gauge field strength operators

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Abstract

We consider circular strings rotating with equal spins $S_1 = S_2 = S$ in two orthogonal planes in AdS_5 and suggest that they may be dual to "long" gauge theory operators built out of self-dual components of gauge field strength. As was found in hep-th/0404187, the one-loop anomalous dimensions of the such gauge-theory operators are described by an anti-ferromagnetic XXX_1 spin chain and scale linearly with length $L \gg 1$. We find that in the case of rigid rotating string both the classical energy E_0 and the 1-loop string correction E_1 depend linearly on the spin S (within the stability region of the solution). This supports the identification of the rigid rotating string with the gauge-theory operator corresponding to the maximal-spin (ferromagnetic) state of the XXX_1 spin chain. The energy of more general rotating and pulsating strings also happens to scale linearly with both the spin and the oscillation number. Such solutions should be dual to other lower-spin states of the spin chain, with the anti-ferromagnetic ground state presumably corresponding to the string pulsating in two planes with no rotation.

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1   Introduction

The study of AdS/CFT duality between string theory in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory uncovered a remarkable connection between dimensions of "long" gauge-theory operators and energies of semiclassical strings in AdS_5 [1, 2, 3, 4, 5, 6, 7, 8, 9] (see also [10, 11, 12, 13] for reviews and references). It is important to investigate other examples of this connection and draw lessons that may apply to less supersymmetric gauge theories, including pure YM theory.

In general, quantum operator dimensions and energies of quantum string states are non-trivial functions of the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$ (or square of string tension) and quantum numbers Q parametrizing them, $\Delta = E = E(\lambda, Q)$. In the limit of large Q the function E should have a gauge-theory perturbative expansion at small λ, $E_{\lambda \ll 1} = a_0(Q) + a_1(Q)\lambda + a_2(Q)\lambda^2 + ...$, and string-theory perturbative expansion at large λ (the semiclassical string expansion corresponds to $\frac{Q}{\sqrt{\lambda}}$ being fixed), $E_{\lambda \gg 1} = \sqrt{\lambda} b_0(\frac{Q}{\sqrt{\lambda}}) + b_1(\frac{Q}{\sqrt{\lambda}}) + \frac{1}{(\sqrt{\lambda})^2} b_2(\frac{Q}{\sqrt{\lambda}}) + ...$

In the case of "locally BPS" operators dual to "fast" strings carrying at least one large angular momentum in S^5 the structure of the two expansions happens to be essentially the same, with the leading (order λ and λ^2) coefficients matching precisely. The general pattern, however, should be an interpolation in λ: terms of different order in large Q expansion may have coefficients which are non-trivial functions of λ and which may have different large and small λ limits.

Here we will be interested in the case of the semiclassical strings moving only within AdS_5 which should thus be dual to the gauge-theory operators which do not carry $SO(6)$ R-charges. This sector of states is very interesting since at least part of it should be common to any gauge theory. Then the "long" operators in this sector may be dual to semiclassical strings in AdS_5, with anomalous dimensions having the same structure as string energies.

The obvious global charges are the two spins (S_1, S_2), i.e. Cartans of $SO(4)$ part of the $SO(2, 4)$ isometry of AdS_5. A generic operator can be represented symbolically as a combination of different orderings of factors in $\mathrm{Tr}(D_+^{S_1} D_*^{S_2} F^M)$ [4] where $D_+ = D_1 + iD_2$, $D_* = D_3 + iD_4$ are covariant derivatives in gauge theory on R^4 and F^M stands for products of gauge field strength components, scalars or spinors (in general, field strength components and spinors may also carry the spins (S_1, S_2)). One may expect that in the semiclassical limit of large quantum numbers (like the spins or string oscillation numbers related to M) the form of the dependence of $\Delta = E$ on quantum numbers may happen to be the same in the small λ (perturbative gauge theory) and the large λ (perturbative string theory) limits.

The basic example [2] is provided by the folded string which is spinning in one plane with its center at rest in the middle of AdS_5 [14], i.e. having $t = \kappa\tau$, $\rho = \rho(\sigma)$,

$\theta = \frac{\pi}{2}$, $\phi_1 = w\tau$, where the metric of AdS_5 is chosen as

$$ds^2 = -\cosh^2 \rho \ dt^2 + d\rho^2 + \sinh^2 \rho \ (d\theta^2 + \sin^2 \theta \ d\phi_1^2 + \cos^2 \theta \ d\phi_2^2) \ . \qquad (1.1)$$

Here $S_1 = S$, $S_2 = 0$ and the string energy at large S is given by [2, 3]

$$E = S + f(\lambda) \ln S + ... \ , \qquad f(\lambda) = a_0 \sqrt{\lambda} + c_1 + \frac{1}{\sqrt{\lambda}} c_2 + ... \ . \qquad (1.2)$$

The same form of S-dependence is found for the perturbative gauge-theory anomalous dimension of the twist-2 ($M = 2$) operators of the structure $\text{Tr}(F D_+^S F)$ where $f(\lambda) = a_1\lambda + a_2\lambda^2 + ...$ (see [15, 16] and references there).

A similar expression $E = S + h(\lambda, n) \ln \frac{S}{n} + ...$ is found [17] for a closed rotating string with n spikes which should be dual to an operator containing n fields and $S \gg n$ covariant derivatives, i.e. $\mathcal{O} \sim \text{Tr}(D_+^{s_1} F ... D_+^{s_n} F) + ...$, $S = \sum_{i=1}^{n} s_i$ (see also [18, 19]).[1] The emerging picture is that the spikes (or singularities in $\rho(\sigma)$) should correspond to field components in $\mathcal{O}$ while the spin should be represented by the total power of the covariant derivative D_+.

An important question is about the structure of operators dual to other semiclassical string states in AdS_5, e.g., pulsating circular strings [20] (see also [21]) or rigid circular strings rotating simultaneously in two orthogonal planes [4, 22]. Here we shall argue that the circular rotating and pulsating strings in AdS_5 which have $S_1 = S_2$ should be dual to gauge-theory operators built out of the self-dual components of the gauge field strength [23], i.e. $\text{Tr}(F_{m_1n_1}^{(+)} ... F_{m_Ln_L}^{(+)})$.[2] Operators with extra covariant derivatives should probably correspond to more general rotating and oscillating string configurations which are no longer circular (i.e. have ρ depending on σ).

The sector of self-dual field-strength operators was studied in [23, 24] where it was found to be closed under renormalization to 1-loop order and the corresponding 1-loop anomalous dimension matrix was identified with the Hamiltonian of an integrable antiferromagnetic XXX_1 spin chain. In contrast to the case of operators with many covariant derivatives and few fields whose anomalous dimension is logarithmically suppressed compared to their canonical dimension $L \sim S \gg 1$, here the anomalous dimension is proportional to the operator length itself, i.e. for large L one should have

$$\Delta = L + \gamma(\lambda) L + ... \ , \qquad \gamma(\lambda) = c_1\lambda + c_2\lambda^2 + ... \ . \qquad (1.3)$$

The 1-loop spin chain contains the ferromagnetic highest-spin state [23, 24] which may be represented as $\mathcal{O}_S \sim \text{Tr}(F_{1+i2,3+i4})^S$ or $\text{Tr}([D_+, D_*])^S$; for this operator $L = 2S$ and

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[1] In this case one is able to establish [17] a qualitative map between coherent operators and semiclassical strings by matching an action for the spikes motion with a coherent state effective action following from the 1-loop gauge theory dilatation operator.

[2] The $SO(4)$ representations may be labeled as (S_L, S_R) where $S_L = \frac{1}{2}(S_1 + S_2)$, $S_R = \frac{1}{2}(S_1 - S_2)$. The self-dual field strength described by $(1, 0)$ representation then corresponds to $S_1 = S_2$.

in $\mathcal{N} = 4$ SYM theory $c_1 = \frac{3}{8\pi^2}$ [24]. Reducing the spin (i.e. contracting some indices of the field strength factors) one gets to the lowest-energy zero-spin (Lorentz scalar) anti-ferromagnetic state $\mathcal{O}_L$ for which $c_1 = \frac{1}{8\pi^2}$ [24].

A natural candidate for a semiclassical string state dual to the ferromagnetic $\mathcal{O}_S$ operator seems to be the rigid circular rotating string solution of [4, 22] which has the required spin quantum numbers $S_1 = S_2 = S$. This string is positioned at fixed radius $\rho = \rho_0$ and has $t = \kappa\tau$, $\phi_1 = w\tau + m\sigma$, $\phi_1 = w\tau - m\sigma$, $\theta = \frac{\pi}{4}$. However, as was found in [4], its classical energy has rather unusual large S behaviour, $E = 2S + n_1(\lambda S)^{1/3} + ...,$ $n_1 = 3 \times 2^{-4/3}$, while the expected asymptotics of anomalous dimension of gauge-theory operators should be S, not $S^{1/3}$. It was conjectured in [4] that perhaps the general interpolating expression may be $E = \Delta = 2S + [p(\lambda) + q(\lambda)S]^{1/3} + ...,$ where $q_{\lambda \gg 1} = \lambda(c_0 + \frac{c_1}{\sqrt{\lambda}} + ...)$ and $q_{\lambda \ll 1} = \lambda(b_1 + b_2\lambda + ...)$. Then at small λ and large S such that $\lambda S \ll 1$ we would get the expected behaviour[3]

$$\Delta = f(\lambda)S + ... , \qquad f(\lambda)_{\lambda \ll 1} = 2 + a_1\lambda + a_2\lambda^2 + ... \qquad (1.4)$$

Here we would like to propose a different way[4] to reconcile the $E(S)$ behaviour on the string theory and the gauge theory sides, supporting the validity of (1.4) also at strong coupling.

As was found in [4, 22], the circular $S_1 = S_2$ solution is unstable if the semiclassical parameter $\mathcal{S} \equiv \frac{S}{\sqrt{\lambda}}$ is bigger than a critical value $\mathcal{S}_{max} \approx 1.17$ (for winding number $m = 1$). This instability is of a "strong" type, i.e. there is an infinite number of bosonic fluctuation modes which are tachyonic.[5] This suggests that one can trust this solution (and, in particular, compute quantum string α' corrections to its energy and try to interpolate to small λ region) only in the stability interval $0 < S < \sqrt{\lambda}\mathcal{S}_{max}$. Since λ is large on the string side, this interval still includes large values of S, and thus we may hope to be able compare to large S, large λ asymptotics of the exact anomalous dimension.

Indeed, by analyzing the behaviour of the classical string energy $E_0 = \sqrt{\lambda}\mathcal{E}(\mathcal{S}) = \sqrt{\lambda}\mathcal{E}(\frac{S}{\sqrt{\lambda}})$ with $\mathcal{S}$ we will find (in section 2) that $\mathcal{E}$ starts as $\sqrt{2\mathcal{S}}$ at small $\mathcal{S}$ (as appropriate for a string in nearly flat space)[6] and then goes as a straight line near

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[3]Note that the gauge-theory expansion assumes that one first expands in λ and *then* takes $S \gg 1$, so one should indeed assume that $\lambda S \ll 1$.

[4]An argument against the above conjecture that $E = \Delta = 2S + [p(\lambda) + q(\lambda)S]^{1/3} + ...$ is that it suggests that perturbative SYM anomalous dimensions may scale as higher powers of the length S at higher loops. However, the linearity of the anomalous dimensions in the length is a very general property which is a consequence of the locality of interactions in the spin chain. It should hold at least until the order of perturbation theory comparable to the length when it may be modified by the effects of winding contributions [25]. We are grateful to K. Zarembo for this remark.

[5]In the case of a similar $J_1 = J_2$ spinning string in S^5 there is only one unstable mode [4].

[6]The small value of $\mathcal{S}$ means that the radial position of the string ρ_0 is also small but then the string is located in the central region where the AdS_5 curvature is small.

$\mathcal{S}$ of order 1, i.e. $S \sim \sqrt{\lambda}\mathcal{S}_{max}$. Moreover, this linear behaviour of E with S in the region close to $\sqrt{\lambda}\mathcal{S}_{max}$ is preserved at the string 1-loop level: computing the leading quantum correction E_1 to the spinning string energy as in [26, 27, 28] we will find that again $E_1 \sim S+$const. This suggests that in general, for $\mathcal{S}$ far enough from zero but within the stability region, one should have

$$E = f(\lambda)S + q(\lambda) + ... , \qquad (1.5)$$

$$f_{\lambda \gg 1} = p_0 + \frac{p_1}{\sqrt{\lambda}} + ... , \qquad h_{\lambda \gg 1} = \sqrt{\lambda}q_0 + q_1 + ... \qquad (1.6)$$

Our results for the tree-level and 1-loop coefficients are (for $m = 1$)

$$p_0 = 2.5 , \qquad p_1 = -0.9 , \qquad q_0 = 0.65 , \qquad q_1 = -2.38 . \qquad (1.7)$$

The matching of the spin quantum numbers and this linear dependence on S thus supports the identification of the gauge-theory operator corresponding (1-loop order) to the ferromagnetic state of the XXX$_1$ spin chain and the $S_1 = S_2$ rigid spinning string solution.

What about other $S_1 = S_2$ string states that should be dual to other $\text{Tr}(F^{(+)}_{m_1 n_1}...F^{(+)}_{m_L n_L})$ operators built out of the self-dual field strength components which have length $L = 2S + \text{N}$ bigger than the spin $2S$? It seems natural to expect that these should still be circular strings; however, reducing the spin may cause the radial coordinate to change with time, i.e. such strings may not be rigid, i.e. they may be oscillating as well as rotating. The lowest 1-loop anomalous dimension anti-ferromagnetic state should then be dual to the $S_1 = S_2 = S \to 0$ limit of such string, i.e. to the pulsating string similar to the one considered in [20] (but now stretched not in one but in two orthogonal planes). As we shall find in section 3, the classical energy of such general $S_1 = S_2$ rotating and pulsating string solution (described by the ansatz $t = t(\tau)$, $\rho = \rho(\tau)$, $\phi_1 = \varphi(\tau) + m\sigma$, $\phi_1 = \varphi(\tau) - m\sigma$, $\theta = \frac{\pi}{4}$) is again a linear function of both the spin S and the oscillation number N in the region where the corresponding semiclassical parameters $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$ and $\mathcal{N} = \frac{\text{N}}{\sqrt{\lambda}}$ are of order 1. In particular, in the limit of $S \to 0$ we should get $E = h(\lambda)\text{N} + ...$, $h(\lambda) = h_0 + \frac{h_1}{\sqrt{\lambda}} + ...$.

More general solutions where ρ depends on σ may correspond to operators with extra covariant derivative insertions. The extreme case will be a folded and bended string rotating in two planes (see sect. 6 of [29] and [22]).

One interesting open question is whether the pulsating solution found in the zero-spin $S_1 = S_2 \to 0$ limit is stable; in general, one expects that pulsation improves stability (this appears to be the case when pulsation is taking place in S^5 direction [30]). It would be important also to explore if there is some analog of a "reduced" sigma model action on the string side and a coherent state action on the XXX$_1$ spin chain side that can be matched as was done in the examples of "fast" string moving

in S^5 [6] and rotating string with spikes in AdS_5 [17]. In the present $S_1 = S_2$ case the spinning strings are, in general, not "fast" (cf. [9]), so presumably one should not drop all time derivatives of the "transverse" string coordinates in taking an appropriate semiclassical limit. The gauge-theory side is described by an *anti*-ferromagnetic spin chain for which it is indeed natural to expect a 2-d Lorentz-invariant coherent state action [23].

We shall start in section 2 with a general discussion of circular strings rotating in two planes in AdS_5. In section 2.1 we shall show that the classical energy of the rigid (non-pulsating) string solution of [4] scales linearly with spin for $\mathcal{S} \sim 1$. In section 2.2 we shall discuss the 1-loop world-sheet correction to the energy of the rigid string and demonstrate that it also scales linearly with spin within the stability region of the solution.

In section 3 we shall consider other rotating and pulsating string solutions with $S_1 = S_2$ and find that their classical energy is also linear both in the spin and the oscillation number.

In Appendix A we shall point out that a solution describing circular string positioned at fixed radial distance ρ and oscillating in the third angle θ of S^3 is actually related by a global $SO(4)$ symmetry transformation to the rigid string solution of [4, 22]. In Appendix B we shall present some details about the derivation of the spectrum of quadratic fluctuations near the rigid rotating string which are used in computation of the 1-loop correction to its energy in section 2.2. Appendix C we shall review the solution of [20] describing string pulsating in one plane of S^3 and find its energy as a function of the oscillation number N following the procedure used for strings pulsating in S^5 in [7]. In Appendix D we apply a similar approach to determine the energy as a function of the spin and oscillation number for more general rotating and pulsating solutions of section 3.

2 Circular $S_1 = S_2$ rotating string in AdS_5

We would like to study a class of circular rotating and pulsating string solutions in AdS_5 with the aim of establishing the dependence of the string energy on the spin and the oscillation number.

Written in terms of 3 complex coordinates

$$\mathrm{Y}_0 \equiv Y_5 + iY_0 \ , \quad \mathrm{Y}_1 \equiv Y_1 + iY_2 \ , \quad \mathrm{Y}_2 \equiv Y_3 + iY_4 \ , \quad \mathrm{Y}_r^* \mathrm{Y}^r = -1 \ , \qquad (2.1)$$

$$\mathrm{Y}_0 = \cosh \rho \ e^{it}, \quad \mathrm{Y}_1 = \sinh \rho \ \sin \theta \ e^{i\phi_1}, \quad \mathrm{Y}_2 = \sinh \rho \ \cos \theta \ e^{i\phi_2} \ , \qquad (2.2)$$

the AdS_5 metric (1.1) is $ds^2 = d\mathrm{Y}_r^* d\mathrm{Y}^r$, where $r, s = 0, 1, 2$ ($\mathrm{Y}^r = \eta^{rs}\mathrm{Y}_s$, with $\eta^{rs} = (-1, 1, 1)$). In this paper we will use conformal gauge. The AdS_5 part of the string

action is then

$$I = \sqrt{\lambda} \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} \, L_{AdS} \,, \qquad (2.3)$$

$$L_{AdS} = -\frac{1}{2} \left[-\cosh^2 \rho \, (\partial t)^2 + (\partial \rho)^2 + \sinh^2 \rho \, ((\partial \theta)^2 + \sin^2 \theta (\partial \phi_1)^2 + \cos^2 \theta (\partial \phi_2)^2) \right] \qquad (2.4)$$

We shall look for circular string solutions of the form (m_1, m_2 are integers)

$$t = t(\tau), \quad \rho = \rho(\tau), \quad \theta = \theta(\tau), \quad \phi_1 = \varphi_1(\tau) + m_1 \sigma, \quad \phi_2 = \varphi_2(\tau) + m_2 \sigma \,. \qquad (2.5)$$

The integrals of motion

$$\mathcal{E} = \cosh^2 \rho \, \dot{t}, \quad \mathcal{S}_1 = \sinh^2 \rho \, \sin^2 \theta \, \dot{\varphi}_1, \quad \mathcal{S}_2 = \sinh^2 \rho \, \cos^2 \theta \, \dot{\varphi}_2 \qquad (2.6)$$

determine the global $SO(2,4)$ charges (energy and two spins)

$$E \equiv \sqrt{\lambda} \mathcal{E} \,, \qquad S_1 \equiv \sqrt{\lambda} \mathcal{S}_1 \,, \qquad S_2 \equiv \sqrt{\lambda} \mathcal{S}_2. \qquad (2.7)$$

One of the two conformal gauge constraints implies

$$m_1 \mathcal{S}_1 + m_2 \mathcal{S}_2 = 0 \,. \qquad (2.8)$$

In this paper we will be interested in the subsector of states for which

$$\mathcal{S}_1 = \mathcal{S}_2 \equiv \mathcal{S} \,, \qquad m_1 = -m_2 \equiv m \,. \qquad (2.9)$$

As explained in the Introduction, they are expected to be related to gauge-theory operators built out of the self-dual components of the gauge field strength. The equations of motion for ρ and θ then are

$$\ddot{\rho} + \mathcal{E}^2 \frac{\sinh \rho}{\cosh^3 \rho} - (\dot{\theta}^2 - m^2) \sinh \rho \cosh \rho - 4\mathcal{S}^2 \frac{\cosh \rho}{\sinh^3 \rho \sin^2 2\theta} = 0 \qquad (2.10)$$

$$\ddot{\theta} + 2 \coth \rho \, \dot{\theta} \dot{\rho} - 8\mathcal{S}^2 \frac{\cos 2\theta}{\sinh^4 \rho \sin^3 2\theta} = 0 \qquad (2.11)$$

They imply conservation of the second conformal gauge constraint

$$\dot{\rho}^2 - \frac{\mathcal{E}^2}{\cosh^2 \rho} + (\dot{\theta}^2 + m^2) \sinh^2 \rho + \frac{4\mathcal{S}^2}{\sinh^2 \rho \sin^2 2\theta} = 0 \qquad (2.12)$$

2.1 Classical energy of rigid rotating string

A particular solution of the above system of equations is found by setting $\rho = \rho_0 =$ const; in this case the string is rotating within S^3 at a fixed radial distance ρ_0 from the center of AdS_5. The remaining τ-dependence of θ can be, in fact, eliminated by a global rotation which sets $\theta = \frac{\pi}{4}$ (we explain this in Appendix A). The resulting solution is

then equivalent to the rigid circular rotating string of [4] in the $SO(4)$ rotated form in which it was presented in [22].

In the cartesian coordinates Y_s the solution looks like [22] ($r_0 \equiv \cosh \rho_0 = \sqrt{1 + 2r_1^2}$)

$$Y_0 = r_0 \, e^{i\kappa\tau} \,, \qquad Y_1 = r_1 \, e^{iw\tau + im\sigma} \,, \qquad Y_2 = r_1 \, e^{iw\tau - im\sigma} \,, \qquad (2.13)$$

where $t = \kappa\tau$ ($\mathcal{E} = \kappa \cosh^2 \rho_0$). One ends up with the following relations

$$\mathcal{E} = \kappa + \frac{2\kappa\mathcal{S}}{\sqrt{\kappa^2 + m^2}} \,, \qquad \kappa^2 \sqrt{m^2 + \kappa^2} = 4m^2 \mathcal{S} \,, \qquad (2.14)$$

$$\mathcal{S} = wr_1^2 \,, \qquad w = \sqrt{m^2 + \kappa^2} \,, \qquad r_1 \equiv \frac{\sinh \rho_0}{\sqrt{2}} = \frac{\kappa}{2m} \,. \qquad (2.15)$$

Here the relation for w comes from the equation for ρ and for r_1 – from the Virasoro condition.[7]

The explicit expression for the energy in terms of the equal spins $\mathcal{S}$ and the winding number m is then

$$\mathcal{E} = \left(\frac{m}{\sqrt{3}A^{2/3}} + \frac{2\mathcal{S}m^{2/3}}{1 + A^{1/3}} \right) \sqrt{A^{2/3} - A^{-1/3} + 1} \,, \qquad (2.16)$$

$$A \equiv -1 + 12\frac{\mathcal{S}}{m^2}(18\mathcal{S} + \sqrt{324\mathcal{S}^2 - 3m^2}) \,. \qquad (2.17)$$

The energy dependence on spin at small $\mathcal{S}$ is the same as in flat space

$$\mathcal{E} = 2\sqrt{m\mathcal{S}} + O(\mathcal{S}^{2/3}) \,. \qquad (2.18)$$

At large $\mathcal{S}$ the energy goes as [4]

$$\mathcal{E} = 2\mathcal{S} + \frac{3}{4}(4m^2\mathcal{S})^{1/3} + \dots \qquad (2.19)$$

However, this solution is not stable [4] for large spin $\mathcal{S}$ (or, equivalently, for large κ or large ρ_0). The region of stability depends on the value of m. For example, for $m = 1$ the solution is stable for $\mathcal{S} \leq 1.17$ (see [4] and Appendix B). The plot of the classical energy in the region of stability is presented in Figure 1. We observe that while for small $\mathcal{S}$ the classical energy E_0 goes as the square root of $\mathcal{S}$ (2.18), in the stability interval $0.5 \leq \mathcal{S} \leq 1.17$ its dependence on the spin is approximately linear

$$E_0 \approx \sqrt{\lambda}(2.5\mathcal{S} + 0.65) = 2.5S + 0.65\sqrt{\lambda} \,, \qquad 0.5 \leq \frac{S}{\sqrt{\lambda}} \leq 1.17 \,. \qquad (2.20)$$

As discussed in the Introduction, since in the semiclassical approximation $S = \sqrt{\lambda}\mathcal{S}$ is still large, this supports the possibility of identification of this classical solution with the gauge-theory operator corresponding to the ferromagnetic state of the 1-loop XXX_1 spin chain of [23].

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[7]It is easy to find a generalization of this solution to non-zero orbital momentum J in a big circle of S^1. The resulting $(E, S1, S_2; J)$ solution is then related by analytic continuation to the circular 3-spin solution $(J_1, J_2, J_3; E)$ in $S^5 \times R_t$ [4].
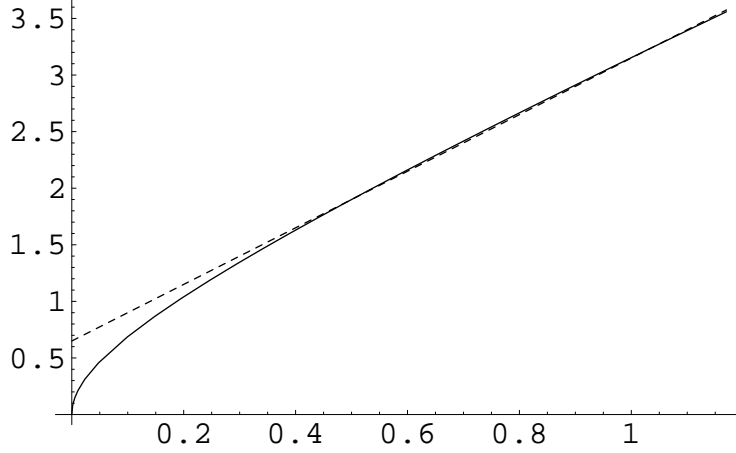
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Figure 1: Plot of the classical energy $\mathcal{E}(\mathcal{S})$ for $m = 1$. The dashed line is $2.5\mathcal{S} + 0.65$.

2.2   One-loop correction to energy of rigid rotating string

To be able to extrapolate the above solution to small λ, large S region one is to be sure that the $AdS_5 \times S^5$ string quantum corrections do not modify the qualitative behaviour of the energy with spin. Below we shall describe the result of the computation of the 1-loop correction to the energy of the rigid circular solution following [4, 26, 27, 28].

The rigid circular rotating string is a "homogeneous" solution for which the fluctuation Lagrangian has constant coefficients [22]. As a result, the spectrum of quadratic fluctuations can be found explicitly (which is a substantial simplification compared to the case of the single-spin folded string solution discussed in [3]).

The three transverse AdS_5 bosonic fluctuations have characteristic frequencies determined by the following equation [4, 22] (see also Appendix B)

$$
\begin{aligned}
0 = {} & \omega^6 - (8m^2 + 10\kappa^2 + 3n^2)\omega^4 + (16m^4 + 40m^2\kappa^2 + 24\kappa^4 + 8n^2\kappa^2 + 3n^4)\omega^2 \\
& - n^2(n^2 - 4m^2)(n^2 - 4m^2 - 2\kappa^2)
\end{aligned}
\tag{2.21}
$$

In addition, there are two free massless ($\omega = \pm n$) AdS_5 bosonic modes and also five free massless S^5 bosonic modes. The condition of stability of the solution is that all bosonic frequencies are real. That implies (we are assuming that $\mathcal{S}$ is positive) [4]

$$
\mathcal{S} \leq \mathcal{S}_{max} , \qquad \mathcal{S}_{max} = \frac{4m + 1}{8m^2}\sqrt{(m + 1)^2 - \tfrac{1}{2}} .
\tag{2.22}
$$

In the previous subsection we have used that $\mathcal{S}_{max}(m = 1) = \frac{5}{8}\sqrt{7\sqrt{2}} \approx 1.17$. The fermionic frequencies are determined (with factor of 4 degeneracy) by the characteristic equation (see Appendix B)

$$
\begin{aligned}
0 = {} & 16\,\omega^4 - 8\left(4\,n^2 + 4\,m^2 + 5\,\kappa^2\right)\omega^2 + 16\,m^4 - 32\,m^2\,n^2 + 16\,n^4 \\
& + 24\,m^2\,\kappa^2 - 8\,n^2\,\kappa^2 + 9\,\kappa^4 ,
\end{aligned}
\tag{2.23}
$$

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with the solution

$$\omega_n^F = \pm\sqrt{n^2 + m^2 + \frac{5}{4}\kappa^2 \pm \sqrt{4\,m^2\,n^2 + m^2\,\kappa^2 + 3\,n^2\,\kappa^2 + \kappa^4}} \tag{2.24}$$

The 1-loop correction to the space-time energy E_1 is given by the sum of the characteristic frequencies (see [3, 4, 26, 28] for details)

$$E_1 = \frac{1}{\kappa}E_{2d} = \frac{1}{2\kappa}\left[\sum_{p=1}^{8}\left(|\omega_{p0}^B| - |\omega_{p0}^F|\right) + \sum_{n=1}^{\infty}\sum_{I=1}^{16}\left(|\omega_{In}^B| - |\omega_{In}^F|\right)\right]. \tag{2.25}$$

Note that here (in contrast to more subtle case discussed in [28]) we can use absolute values of the frequencies since half of the frequencies are positive and half negative (both bosonic and fermionic characteristic equations are expressed in terms of ω^2). It remains only to compute the sum in (2.25). The sum is convergent as follows from the large n asymptotics of the non-trivial bosonic and fermionic frequencies

$$\omega_{1,2n}^B = |n| \pm \sqrt{3\kappa^2 + 4m^2} + \frac{\kappa^2}{2|n|} + O(\frac{1}{n^2}), \qquad \omega_{3n}^B = |n| + \frac{\kappa^2}{|n|} + O(\frac{1}{n^2}), \tag{2.26}$$

$$\omega_{1,2n}^F = |n| \pm \frac{1}{2}\sqrt{3\kappa^2 + 4m^2} + \frac{\kappa^2}{4|n|} + O(\frac{1}{n^2}) \tag{2.27}$$

Taking into account the degeneracies and including the contribution of the bosonic frequencies in the S^5 directions we then get

$$\sum_I \omega_{In}^B = 16|n| + \frac{4\kappa^2}{|n|} + O(\frac{1}{n^2}), \qquad \sum_I \omega_{In}^F = 16|n| + \frac{4\kappa^2}{|n|} + O(\frac{1}{n^2}), \tag{2.28}$$

which confirms the convergence of the sum in (2.25).

Due to the rather complicated form of the explicit solutions for the characteristic frequencies we are unable to perform the sum in (2.25) analytically. However, it is straightforward to evaluate it numerically in the region of stability of the solution when the frequencies and thus E_1 are real. For $m = 1$, the resulting one-loop correction E_1 is plotted in Figure 2 as a function of $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$. We find that there is again an interval within the stability region (from around 0.4 to 1.15) where E_1 can be well approximated by a straight line

$$E_1 \approx -0.9\frac{S}{\sqrt{\lambda}} - 2.38. \tag{2.29}$$

The total energy is then $E = E_0 + E_1 + ..$, where E_0 and E_1 are given by (2.20) and (2.29). This supports the conjecture that the general large-S expression for the string energy that may be possible to extrapolate to weak coupling is given by (1.5).
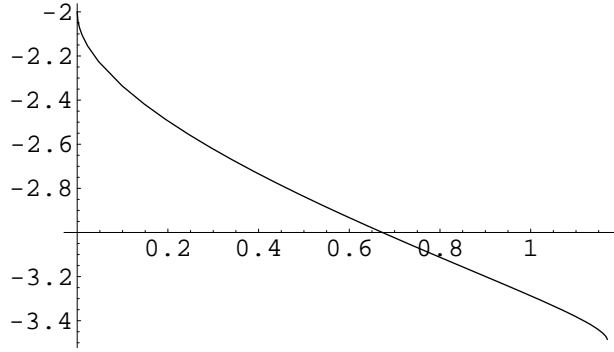
Figure 2: One-loop correction to energy E_1 as a function of $\mathcal{S}$

3   Rotating and pulsating string

Let us now return to the equations (2.10),(2.11), (2.12) and look for more general circular rotating and pulsating solutions with $S_1 = S_2$. Our conjecture is that they may be dual to other (lower-spin) operators built out of self-dual gauge-field components.

As was mentioned above, having ρ=const while θ changing with time does not lead to a new solution. Here we shall consider the alternative case when θ remains fixed at $\frac{\pi}{4}$ while ρ is allowed to change in time. This will generalize the pulsating solution discussed in [20] where the string was lying only in one plane (i.e. had $\theta = \frac{\pi}{2}$) and was not rotating (we shall review this solution in Appendix C).

The solution with $S_1 = S_2$ we are interested in is described by (2.5) with $\varphi_1 = \varphi_2$ and fixed θ, i.e. by the following ansatz

$$t = t(\tau) , \quad \rho = \rho(\tau), \quad \theta = \frac{\pi}{4}, \quad \phi_1 = \varphi(\tau) + m\sigma, \quad \phi_2 = \varphi(\tau) - m\sigma . \quad (3.1)$$

The conserved charges are

$$\mathcal{E} = \cosh^2 \rho \ \dot{t} , \qquad \mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S} = \frac{1}{2} \sinh^2 \rho \ \dot{\varphi} . \quad (3.2)$$

The conformal constraint gives the equation of motion for ρ

$$\dot{\rho}^2 + V(\rho) = 0 , \qquad V \equiv -\frac{\mathcal{E}^2}{\cosh^2 \rho} + \frac{4\mathcal{S}^2}{\sinh^2 \rho} + m^2 \sinh^2 \rho . \quad (3.3)$$

This may be interpreted as an equation for a particle moving in a potential which is growing to infinity both at $\rho \to 0$ and $\rho \to \infty$ and having a minimum in between. The coordinate $\rho(\tau)$ thus oscillates between a minimal amd maximal value.

Following the discussion of pulsating solutions in S^5 in [7] we can compute the oscillation number (which should take integer values in quantum theory) as

$$N = \frac{\sqrt{\lambda}}{2\pi} \oint d\rho \ p_\rho = \frac{\sqrt{\lambda}}{\pi} \int_{\rho_{min}}^{\rho_{max}} d\rho \sqrt{\frac{\mathcal{E}^2}{\cosh^2 \rho} - \frac{4\mathcal{S}^2}{\sinh^2 \rho} - m^2 \sinh^2 \rho} . \quad (3.4)$$

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Changing the variable to $x = \sinh \rho$ we get

$$N = \frac{\sqrt{\lambda}}{\pi} \int_{\sqrt{R_2}}^{\sqrt{R_3}} \frac{dx}{1+x^2} \sqrt{\mathcal{E}^2 - \frac{4\mathcal{S}^2(1+x^2)}{x^2} - m^2 x^2(1+x^2)} \qquad (3.5)$$

where R_2, R_3 are the two positive roots of the cubic polynomial

$$f(y) = m^2 y^2 (1+y) + y(4\mathcal{S}^2 - \mathcal{E}^2) + 4\mathcal{S}^2 , \qquad y \equiv x^2 . \qquad (3.6)$$

The third root (R_1) is negative.

One can show (see Appendix D) that in order for this solution to exist the function inside the square root in (3.5) must be positive. This gives a condition on the energy, spin and the winding number m

$$\mathcal{E}^2 \geq \max \left\{ 4\mathcal{S}^2, \; h(m,\mathcal{S}) \right\} \qquad (3.7)$$

where $h(m,\mathcal{S})$ is the maximal root of the polynomial $g(z)$ (which has at least one real root)

$$g(z) = -4z^3 + z^2(48\mathcal{S}^2 - m^2) - 16\mathcal{S}^2 z(12\mathcal{S}^2 - 5m^2) + 16\mathcal{S}^2(4\mathcal{S}^2 + m^2)^2 . \qquad (3.8)$$

The asymptotic dependence of the energy on the oscillation number N and spin is worked out in Appendix D. For small $\mathcal{S}$ and $\mathcal{N} \equiv \frac{N}{\sqrt{\lambda}}$ the energy dependence is like in flat space

$$E = 2\lambda^{1/4}\sqrt{mL} + ... , \qquad L \equiv N + S . \qquad (3.9)$$

For large $\mathcal{N}$ but small $\mathcal{S}$ we get

$$E = 2L + d_1 \lambda^{1/4}\sqrt{L} + d_2\sqrt{\lambda} + d_3 \frac{\lambda^{3/4}}{\sqrt{L}} + O(L^{-3/2}) , \qquad (3.10)$$

where

$$L \equiv N + S , \qquad d_1 \approx 0.359 m^{3/2} , \qquad d_2 \approx 0.032 m^3 , \qquad d_3 \approx -0.059 m^{5/2} . \qquad (3.11)$$

Starting with the $O(L^{-3/2})$ term, the coefficients will depend on the spin S explicitly, i.e. not only through L. For large $\mathcal{S}$ but small $\mathcal{N}$ the energy is given by (for $m = 1$)

$$E = 2S + v_1 N + v_2(\lambda S)^{1/3} + O(S^{-1/3}) , \qquad v_1 = 1.83 , \quad v_2 = 0.45 . \qquad (3.12)$$

Let us now consider the dependence of E on N and S in the intermediate region of $\mathcal{N}$ and $\mathcal{S}$ of order 1 (for $m = 1$), where, as in the rigid string case, the solution is expected to be stable (pulsation, in general, should improve stability). Let us consider the region $0.4 \leq \mathcal{S} \leq 1$. According to (3.7), this means that $\mathcal{E}$ is at least greater than $2\mathcal{S}$. For each value of the spin $\mathcal{S}$ we can determine the minimum energy $\mathcal{E}_{min}$ from (3.7). We computed numerically the oscillation number $\mathcal{N} = N/\sqrt{\lambda}$ using (3.5) and

plotted it as a function of $\mathcal{E}$ for several values of $\mathcal{S}$ from $\mathcal{E} = \mathcal{E}_{min}$ to $\mathcal{E} = 5$. Near $\mathcal{E}_{min}$ we got the expected parabolic dependence ($\mathcal{N} \sim \frac{\mathcal{E}^2}{4} - \mathcal{S}$) as in flat space since there the string is close to the origin of AdS_5. The string is oscillating near the radial point ρ_* which is not far from 0. The results are presented in the Table below.

$\mathcal{S}$	$\mathcal{E}_{min}$	$\mathcal{N} = \mathcal{N}(\mathcal{E})$	ρ_*
0.4	1.629	-0.727+0.401 $\mathcal{E}$	0.686
0.6	2.161	-0.918+0.401 $\mathcal{E}$	0.786
0.8	2.665	-1.106+0.4006 $\mathcal{E}$	0.86
1	3.153	-1.292+ 0.4002 $\mathcal{E}$	0.919

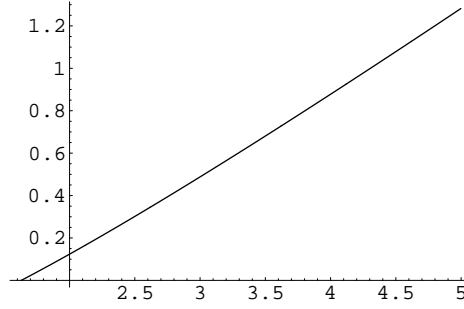The plot of $\mathcal{N}(\mathcal{E})$ for $\mathcal{S} = 0.4$ is given in Figure 3.

Figure 3: $\mathcal{N}(\mathcal{E})$ for $\mathcal{S} = 0.4$

We observe that the slope of the linear function $\mathcal{N}(\mathcal{E})$ does not depend on $\mathcal{S}$, i.e. $\mathcal{N} \approx a(\mathcal{S}) + 0.4\mathcal{E}$. For $\mathcal{N} = 0$ and $\mathcal{S}$ within the range $0.4 \leq \mathcal{S} \leq 1$ we should reproduce the straight line for $\mathcal{E} = \mathcal{E}(\mathcal{S})$ found in the rigid string case. The plot of $a(\mathcal{S})$ is well approximated by the straight line $a(\mathcal{S}) = -0.248 - 1.017\mathcal{S}$. We conclude then that

$$\mathcal{N} = -0.248 - 1.017\mathcal{S} + 0.4\mathcal{E} , \tag{3.13}$$

so that the classical energy of rotating and pulsating string has the linear form in the region where $N \sim \mathcal{S} \sim \sqrt{\lambda}$

$$E_0 \approx 2.5N + 2.54S + 0.62\sqrt{\lambda} . \tag{3.14}$$

For $N = 0$ this is in reasonable agreement with the rigid string expression (2.20).

Written in terms of the length $L = S + N$ and the spin the energy is $E_0 \approx 2.5L + 0.04S + 0.62\sqrt{\lambda}$, i.e. grows with spin for fixed length as does the anomalous dimension on the gauge-theory side [24]. Note also that the rigid string ($N = 0$) has lowest energy for given spin S – adding oscillations increases the energy. This is in

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qualitative agreement with the gauge-theory interpretation where $\mathrm{Tr}F^S$ operator has lowest anomalous dimension for given spin.

Repeating the above discussion in the case of $\mathcal{S} = 0$ we find a similar linear dependence of E on N $\sim \sqrt{\lambda}$. This supports the conjecture that this pulsating string state should be dual to the gauge-theory operator represented by the anti-ferromagnetic vacuum of the XXX_1 spin chain [23, 24].

Acknowledgments

Appendix A: String with $\rho =$const and oscillating θ

Here we would like to show that the spinning string solution with constant ρ which exists if $S_1 = S_2$ and with θ changing with time is actually equivalent to the solution with $\theta = \frac{\pi}{4}$ we considered in section 2.2. For $t = \kappa\tau$ the equation of motion for ρ and the conformal constraint give

$$\kappa^2 = 2m^2 \sinh^2 \rho_0 \, , \qquad \dot{\theta}^2 = w^2 - \frac{g^2}{\sin^2 2\theta} \tag{A.1}$$

where

$$w^2 = m^2 + \kappa^2 \, , \qquad g^2 = \frac{16m^4 \mathcal{S}^2}{\kappa^4} \, . \tag{A.2}$$

Then θ can oscillate in the range

$$\theta_{min} \leq \theta \leq \frac{\pi}{2} - \theta_{min} \equiv \theta_{max} \, , \qquad \theta_{min} \equiv \frac{1}{2} \arcsin \frac{g}{w} \, . \tag{A.3}$$

The explicit form of the solution for θ is

$$\cos 2\theta = \sqrt{1 - \frac{g^2}{w^2}} \, \sin 2w\tau \tag{A.4}$$

As τ increases from zero to $w\tau = \frac{\pi}{4}$, θ decreases from $\frac{\pi}{2}$ to θ_{min}. The solutions for the two angles ϕ_1, ϕ_2

$$\varphi_1 = \arctan\left(\frac{w}{g} \tan w\tau - \sqrt{\frac{w^2}{g^2} - 1}\right) , \qquad \varphi_2 = \arctan\left(\frac{w}{g} \tan w\tau + \sqrt{\frac{w^2}{g^2} - 1}\right) . \tag{A.5}$$

Expressing this solution in cartesian coordinates (2.2) we get

$$\text{Y}_1 = \frac{\sinh \rho_0}{\sqrt{2}w} e^{im\sigma} \left(a e^{iw\tau} + b e^{-iw\tau} \right) , \qquad \text{Y}_2 = \frac{\sinh \rho_0}{\sqrt{2}w} e^{-im\sigma} \left(c e^{iw\tau} + d e^{-iw\tau} \right) \quad \text{(A.6)}$$

where

$$a, b = \frac{1}{2}(g \pm w - i\sqrt{w^2 - g^2}) , \qquad c = \bar{a} , \quad d = \bar{b} , \quad ad + bc = 0 . \quad \text{(A.7)}$$

The corresponding oscillation number in θ direction (i.e. the action variable conjugate to angle θ which for periodic motion is quantized in semiclassical quantization)[8] is (here $p_\theta = \sinh^2 \rho_0 \, \dot{\theta}$)

$$\text{N} = \frac{\sqrt{\lambda}}{2\pi} \oint d\theta \, p_\theta = \frac{\sqrt{\lambda}\kappa^2}{2\pi m^2} \int_{\theta_{min}}^{\theta_{max}} d\theta \sqrt{w^2 - \frac{g^2}{\sin^2 2\theta}} = \frac{\sqrt{\lambda}\kappa^2}{4m^2}(w - g) . \quad \text{(A.8)}$$

When $w = g$ we get, as expected, N = 0: then $\theta = \frac{\pi}{4}$, i.e. there are no oscillations. We can also rewrite N as

$$\text{N} = \frac{\sqrt{\lambda}\kappa^2 \sqrt{m^2 + \kappa^2}}{4m^2} - S , \quad \text{(A.9)}$$

and recalling that the energy is

$$\mathcal{E} = \kappa + \frac{\kappa^3}{2m^2} , \quad \text{(A.10)}$$

we conclude that it depends on N and S only through their sum $L = \text{N} + S$. This suggests that this pulsating solution is not a new one but a rotated version of the rigid solution with $\theta = \frac{\pi}{4}$. Indeed, starting with (A.6) one can show that by an $SO(4)$ rotation this solution can be transformed into the rigid rotating solution (2.13) (this $SO(4)$ rotation implies also the redefinition of the spin $S + \text{N} \to S$).

Appendix B: Fluctuations near rigid rotating string

Here we provide some details for the computation of 1-loop correction to the energy of the rotating string solution in section 2.2. Let us first review the the form of the bosonic fluctuation Lagrangian following [22]. Since the string is moving entirely in AdS_5 the fluctuations in S^5 directions are trivial (5 massless 2d bosons). The fluctuation Lagrangian in the AdS_5 directions near the solution (2.13) is

$$\tilde{L}_{AdS} = -\frac{1}{2} \, \partial_a \tilde{Y}^r \partial^a \tilde{Y}_r^* - \frac{1}{2}\tilde{\Lambda} \, \tilde{Y}^r \tilde{Y}_r^* , \qquad \sum_{r=0}^{2} (Y^r \tilde{Y}_r^* + Y_r^* \tilde{Y}^r) = 0 . \quad \text{(B.1)}$$

---

[8]The string motion is described by an integrable 1-d Neumann model [29, 22], and the energy, spins and the oscillation number are related to its integrals of motion.

Writing the fluctuation fields as

$$\tilde{Y}_0 = (g_0 + if_0)e^{i\kappa\tau}, \quad \tilde{Y}_1 = (g_1 + if_1)e^{iw\tau+im\sigma}, \quad \tilde{Y}_2 = (g_2 + if_2)e^{iw\tau-im\sigma} , \quad (\text{B.2})$$

the above constraint gives (see (2.13)) $g_0 = \frac{r_1(g_1+g_2)}{r_0}$. Then the fluctuation Lagrangian for the five real fields g_1, g_2, f_0, f_1, f_2 takes the form

$$\begin{aligned}
\tilde{L} &= -\frac{1}{2}[\dot{f}_0^2 - f_0'^2 + 4\kappa\frac{r_1}{r_0}\dot{f}_0(g_1 + g_2)] - \frac{r_1^2}{2r_0^2}\left[(\dot{g}_1 + \dot{g}_2)^2 - (g_1' + g_2')^2\right] \\
&+ \frac{1}{2}\left(\dot{g}_1^2 + \dot{f}_1^2 + \dot{g}_2^2 + \dot{f}_2^2 - g_1'^2 - f_1'^2 - g_2'^2 - f_2'^2\right) \\
&+ 2\left(wg_1\dot{f}_1 - mg_1f_1' + wg_2\dot{f}_2 + mg_2f_2'\right) .
\end{aligned} \quad (\text{B.3})$$

This determines the bosonic characteristic frequencies (by expanding the fields as $\sim e^{in\sigma+iw\tau}$ and finding the condition of consistency of the resulting linear system of equations). We find that there are two massless modes which decouple (in conformal gauge their contribution is effectively cancelled by the contribution of the ghosts), and for the remaining coupled three directions we obtain the characteristic equation [4, 22] (2.21), or $f_n(\omega^2) = 0$, where

$$\begin{aligned}
f_n(x) &= x^3 - (8m^2 + 10\kappa^2 + 3n^2)x^2 + (16m^4 + 40m^2\kappa^2 + 24\kappa^4 + 8n^2\kappa^2 + 3n^4)x \\
&- n^2(n^2 - 4m^2)(n^2 - 4m^2 - 2\kappa^2) .
\end{aligned} \quad (\text{B.4})$$

Following [4], we conclude that the stability condition when all ω's are real is

$$f_n(0) = -n^2(n^2 - 4m^2)(n^2 - 4m^2 - 2\kappa^2) \leq 0 . \quad (\text{B.5})$$

Here $f_0(0) = f_{2m}(0) = 0$, $f_1(0), ..., f_{2m-1}(0) < 0$, $f_{2m+1}(0) = -(2m+1)^2(4m+1)(4m+1-2\kappa^2)$. The stability condition is then

$$\kappa^2 < 2m + \frac{1}{2} . \quad (\text{B.6})$$

Expressed in terms of $\mathcal{S}$ it is $\mathcal{S} \leq \mathcal{S}_{max}$, with $\mathcal{S}_{max} = \frac{4m+1}{8m^2}\sqrt{(m+1)^2 - \frac{1}{2}}$.

The fermionic fluctuations are described by the quadratic part of the $AdS_5 \times S^5$ superstring Lagrangian evaluated on the bosonic solution (2.13) (see [31, 3, 4, 26] for details).

$$L_F = i\left(\eta^{ab}\delta^{IJ} - \epsilon^{ab}s^{IJ}\right)\bar{\theta}^I\rho_a D_b\theta^J , \qquad \rho_a \equiv \Gamma_A e_a^A , \qquad e_a^A \equiv E_\mu^A(\mathcal{X})\partial_a\mathcal{X}^\mu , \quad (\text{B.7})$$

where $I, J = 1, 2$, $s^{IJ} = \text{diag}(1, -1)$, ρ_a are projections of the ten-dimensional Dirac matrices and $\mathcal{X}^\mu$ are the coordinates of the AdS_5 space for $\mu = 0, 1, 2, 3, 4$ (i.e.

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$t, \rho, \theta, \phi_1, \phi_2$) and the coordinates of S^5 for $\mu = 5, 6, 7, 8, 9$. The covariant derivative is given by

$$D_a \theta^I = \left(\delta^{IJ} D_a - \frac{i}{2} \epsilon^{IJ} \Gamma_* \rho_a \right) \theta^J , \qquad \Gamma_* \equiv i\Gamma_{01234} , \quad \Gamma_*^2 = 1 , \qquad \text{(B.8)}$$

where $D_a = \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}$, $\omega_a^{AB} \equiv \partial_a \mathcal{X}^\mu \omega_\mu^{AB}$. Fixing the κ-symmetry by the same condition as in [26] $\theta^1 = \theta^2 = \theta$ one gets

$$L_F = -2i\bar{\theta}D_F\theta , \qquad D_F = -\rho^a D_a - \frac{i}{2}\epsilon^{ab}\rho_a \Gamma_* \rho_b . \qquad \text{(B.9)}$$

Explicitly, one finds

$$
\begin{aligned}
D_F = \quad & (\kappa r_0 \Gamma_0 + w r_1 \Gamma_3 + w r_1 \Gamma_4) \\
& \times (\partial_\tau - \frac{1}{\sqrt{2}}\kappa r_1 \Gamma_{10} - \frac{w r_0}{2\sqrt{2}}\Gamma_{13} - \frac{w r_0}{2\sqrt{2}}\Gamma_{14} - \frac{w}{2\sqrt{2}}\Gamma_{23} + \frac{w}{2\sqrt{2}}\Gamma_{24}) \\
& - (m r_1 \Gamma_3 - m r_1 \Gamma_4)(\partial_\sigma - \frac{m r_0}{2\sqrt{2}}\Gamma_{13} + \frac{m r_0}{2\sqrt{2}}\Gamma_{14} - \frac{m}{2\sqrt{2}}\Gamma_{23} - \frac{m}{2\sqrt{2}}\Gamma_{24}) \\
& - (m\kappa r_0 r_1 \Gamma_3 \Gamma_0 + 2mw r_1^2 \Gamma_3 \Gamma_4 - m\kappa r_0 r_1 \Gamma_4 \Gamma_0)\Gamma_{01234} \qquad \text{(B.10)}
\end{aligned}
$$

Expanding the fields in terms of $e^{i\omega\tau + in\sigma}$ can show that the $\det(D_F) = 0$ condition leads to the characteristic equation (2.23). In the $4 + 6$ dimensional representation of the Dirac matrices one can use the six dimensional Γ-matrices for the actual computation of the characteristic equation. In (2.23) there is an extra factor of 4 degeneracy making up the total 16 of fermionic frequencies.

Appendix C: String pulsating in one plane

Here we shall review the pulsating AdS_5 solution of [20] using the conformal gauge as in the case of the pulsating solutions in S^5 discussed in [7]. We shall start with (2.3) and consider the case where the string is pulsating in one plane:

$$t = t(\tau) , \quad \rho = \rho(\tau), \quad \theta = \frac{\pi}{2}, \quad \phi_1 = m\sigma, \quad \phi_2 = 0 \qquad \text{(C.1)}$$

Then the only non-zero global $SO(2,4)$ charge is the energy

$$\mathcal{E} = \cosh^2 \rho \, \dot{t} . \qquad \text{(C.2)}$$

The conformal gauge constraint gives the equation of motion for ρ

$$\dot{\rho}^2 - \frac{\mathcal{E}^2}{\cosh^2 \rho} + m^2 \sinh^2 \rho = 0 , \qquad \text{(C.3)}$$

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which is the same as the $\mathcal{S} = 0$ limit pf (3.3). As in [7] the strategy is to compute the oscillation number N and then express the energy in terms of it. By definition (here $p_\rho = \dot\rho$)

$$N = \frac{\sqrt{\lambda}}{2\pi} \oint d\rho \, p_\rho = \frac{\sqrt{\lambda}}{\pi} \int_0^{\rho_{max}} d\rho \sqrt{\frac{\mathcal{E}^2}{\cosh^2 \rho} - m^2 \sinh^2 \rho} \qquad (C.4)$$

Changing the variable to $x = \sinh \rho$ we get

$$N = \frac{\sqrt{\lambda}}{\pi} \int_0^{\sqrt{R}} \frac{dx}{1 + x^2} \sqrt{\mathcal{E}^2 - m^2 x^2 (1 + x^2)} \,, \qquad R = \frac{-m + \sqrt{m^2 + 4\mathcal{E}^2}}{2m} \,. \qquad (C.5)$$

Then

$$\frac{\partial N}{\partial m} = -\frac{m\sqrt{\lambda}}{\pi} \int_0^{\sqrt{R}} dx \frac{x^2}{\sqrt{\mathcal{E}^2 - m^2 x^2 (1 + x^2)}} \qquad (C.6)$$

can be expressed in terms of the elliptic integrals

$$\frac{\partial N}{\partial m} = \frac{1}{2\pi} \sqrt{\frac{\lambda a_-}{2m}} \frac{i a_+}{\mathcal{E}} \left(\mathbf{K} \left[\frac{a_-}{a_+} \right] - \mathbf{E} \left[\frac{a_-}{a_+} \right] \right) \,, \qquad a_\pm \equiv m \pm \sqrt{m^2 + 4\mathcal{E}^2} \,. \qquad (C.7)$$

This representation is useful since it allows one to obtain convergent expansions for N at large and small energies.

Expanding in large $\mathcal{E}$ and integrating from 0 to m we get

$$N(E, m) = N_0(E) + \frac{1}{4} c_1 \lambda^{1/4} \sqrt{2mE} + O(E^0) \,, \qquad (C.8)$$

where

$$c_1 = \frac{4\sqrt{2}}{\pi} \left(\mathbf{E}[-1] - \mathbf{K}[-1] \right) \approx 1.078 \,. \qquad (C.9)$$

The constant of integration N_0 can be determined from the $m = 0$ integral[9]

$$N_0 = \frac{\sqrt{\lambda}\mathcal{E}}{\pi} \int_0^\infty \frac{dx}{1 + x^2} = \frac{1}{2} E \,. \qquad (C.10)$$

Then inverting the relation between N and E we end up with

$$E = 2N + c_1 \lambda^{1/4} \sqrt{mN} + O(N^0) \,, \qquad (C.11)$$

which is the same expression as found in [20].

Expanding (C.7) at small $\mathcal{E}$ gives

$$\frac{\partial N}{\partial m} = \sqrt{\lambda} \left[\frac{\mathcal{E}^2}{4m^2} - \frac{15\mathcal{E}^4}{32m^4} + O \left(\frac{\mathcal{E}^6}{m^6} \right) \right] \,. \qquad (C.12)$$

---

[9]The $m = 0$ limit corresponds to a massless geodesic going from $\rho = 0$ to $\rho = \infty$.

We observe that the right hand side is regular at $m \to \infty$. Integrating from m to ∞ and setting $\mathrm{N}(\mathcal{E}, m = \infty) = 0$ we get

$$\mathrm{N} = \frac{E^2}{4m}\lambda^{-1/2} - \frac{45}{32}\frac{E^4}{m^3}\lambda^{-3/2} + O\left(\frac{E^6}{m^5}\lambda^{-5/2}\right) , \tag{C.13}$$

and thus

$$E = 2\lambda^{1/4}\sqrt{m\mathrm{N}} + O(\mathrm{N}^{3/2}) . \tag{C.14}$$

This is the flat-space dependence which is expected in the small-energy limit where the string is pulsating near the center of AdS_5.

Appendix D: String rotating and pulsating in two planes

Here we shall derive the asymptotic expansions of the energy of the general rotating and pulsating solution discussed in section 3.

Let us first analyze the conditions when this solution exists. We need to have the expression inside the square root in (3.4) positive, i.e. where $f(y) \leq 0$ ($f(y)$ was defined in (3.6), $y = \sinh^2 \rho$). The extrema of f, i.e. the solutions of $f'(y) = 0$ are

$$y_\pm = \frac{-m \pm \sqrt{m^2 + 3(\mathcal{E}^2 - 4\mathcal{S}^2)}}{3m} \tag{D.1}$$

It follows that $f(0) = 4\mathcal{S}^2 > 0$, and that $f(y) \to -\infty$ as $y \to -\infty$, with $f(y) \to \infty$ as $y \to \infty$. To have $f(y) \leq 0$ in a region inside the allowed range of values $y > 0$ we need $y_\pm$ to be real. Since $y_- < 0$ to have $f(y) \leq 0$ we need $y_+ > 0$. This requires $\mathcal{E}^2 \geq 4\mathcal{S}^2$. Then $f(y_-) > 0$ and so to have the solution we need $f(y_+) \leq 0$, where

$$f(y_+) = \frac{1}{27m}\left[2m^3 + 9m(8\mathcal{S}^2 + \mathcal{E}^2) - [2m^2 - 6(4\mathcal{S}^2 - \mathcal{E}^2)]\sqrt{m^2 + 3(\mathcal{E}^2 - 4\mathcal{S}^2)}\right] \tag{D.2}$$

From the form of $f(y_+)$ we conclude that to have a solution we need $g(z) \leq 0$ ($z \equiv \mathcal{E}^2$) for some z within the region $z > 0$, where $g(z)$ was defined in (3.8). We have $g(0) > 0$ and $g(z) \to -\infty$ as $z \to \infty$, $g(z) \to \infty$ as $z \to -\infty$. Then $g(z) = 0$ has at least one real solution. We want $g(z) \leq 0$ in a region within $z > 0$. Therefore, we need $z \equiv \mathcal{E}^2 \geq h(m, \mathcal{S})$ where $h(m, \mathcal{S}$ is the maximal root of $g(z)$ (if the maximal root is negative we just need $z \geq 0$). In addition, we have to satisfy $\mathcal{E}^2 \geq 4\mathcal{S}^2$, so we conclude that the solution exists if

$$\mathcal{E}^2 \geq \max\left\{4\mathcal{S}^2, h(m, \mathcal{S})\right\} \tag{D.3}$$

Note that for small or large $\mathcal{S}$ we have $h(m, \mathcal{S}) > 4\mathcal{S}^2$.

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To study the small energy behavior $\mathcal{E} \to 0$, we thus need also $\mathcal{S} \to 0$. In this limit $h(m,\mathcal{S}) = 4m\mathcal{S} + 8\mathcal{S}^2 + O(\mathcal{S}^3)$, so we are to require $\mathcal{E}^2 \geq 4m\mathcal{S}$. The case of $\mathcal{E}^2 = 4m\mathcal{S}$ corresponds to ρ constant, i.e. to no pulsation or $\mathcal{N} = 0$, when, as in (2.18), we get

$$\mathcal{E} = 2\sqrt{m\mathcal{S}} + 2m^{-1/2}\mathcal{S}^{3/2} + O(\mathcal{S}^{5/2}) \ . \tag{D.4}$$

More generally, one finds from (3.5) that in the limit of small $\mathcal{E}$ and $\mathcal{S}$

$$E = 2\lambda^{1/4}\sqrt{m(S + \mathrm{N})} + \dots \ . \tag{D.5}$$

Again, this is the flat-space behavior, which is expected in this case since the string is located near the origin of AdS_5.

In the large energy limit we can expand $h(m,\mathcal{S})$ at large $\mathcal{S}$. In the case of the equality $\mathcal{E}^2 = h(m,\mathcal{S})$, which again corresponds to constant ρ, i.e. $\mathcal{N} = 0$, we reproduce the asymptotics of the rigid string case (2.19)

$$\mathcal{E} = 2\mathcal{S} + \frac{3}{4}(4m^2)^{1/3}\mathcal{S}^{1/3} + O(\mathcal{S}^{-1/3}) \tag{D.6}$$

Let us now analyze the large energy behavior starting from (3.5) or

$$\frac{\partial \mathrm{N}}{\partial m} = -\frac{m\sqrt{\lambda}}{\pi} \int_{\sqrt{R_2}}^{\sqrt{R_3}} dx \frac{x^2}{\sqrt{\mathcal{E}^2 - \frac{4\mathcal{S}^2(1+x^2)}{x^2} - m^2 x^2 (1 + x^2)}} \ . \tag{D.7}$$

This integral can be expressed in terms of the elliptic integrals $\mathbf{E}$,$\mathbf{F}$

$$\frac{\partial \mathrm{N}}{\partial m} = \frac{m\sqrt{\lambda}}{\pi\sqrt{R_3 - R_1}} \left((R_1 - R_3)\mathbf{E}\left[\frac{R_2 - R_3}{R_1 - R_3}\right] - R_1 \mathbf{F}\left[\frac{R_2 - R_3}{R_1 - R_3}\right] \right) \ . \tag{D.8}$$

The large energy behavior of the roots R_1,R_2,R_3 of $f(y)$ in (3.6) is

$$R_{1,3} = \mp\frac{\mathcal{E}}{m} - \frac{1}{2} \mp \frac{m^2 - 16\mathcal{S}^2}{8m\mathcal{E}} + O(\mathcal{E}^{-2}), \qquad R_2 = \frac{4\mathcal{S}^2}{\mathcal{E}^2} + O(\mathcal{E}^{-4}) \ . \tag{D.9}$$

Expanding for large energy $\mathcal{E}$ we obtain

$$\frac{\partial \mathrm{N}}{\partial m} = \frac{\sqrt{\lambda}}{\pi\sqrt{2}}\left[k_1\sqrt{m\mathcal{E}} + \frac{k_2 m^{3/2}}{4\sqrt{2}\sqrt{\mathcal{E}}} - \frac{k_1\sqrt{m}}{32\mathcal{E}^{3/2}}(3m^2 - 32\mathcal{S}^2) + O(\mathcal{E}^{-5/2})\right] , \tag{D.10}$$

$$k_1 \equiv \mathbf{K}\left[\frac{1}{2}\right] - 2\mathbf{E}\left[\frac{1}{2}\right] \ , \qquad k_2 \equiv \mathbf{K}\left[\frac{1}{2}\right] \tag{D.11}$$

Integrating over m we obtain N

$$\mathrm{N}(\mathcal{E},\mathcal{S}) = \mathrm{N}_0(\mathcal{E},\mathcal{S}) + \frac{\sqrt{2\lambda}m^{3/2}}{\pi}\left[\frac{2k_1\sqrt{\mathcal{E}}}{9} + \frac{k_2 m}{20\sqrt{\mathcal{E}}} - \frac{k_1\left(\frac{3}{7}m^2 - \frac{32}{3}\mathcal{S}^2\right)}{32\mathcal{E}^{3/2}}\right] + \dots \tag{D.12}$$

The integration constant $N_0(\mathcal{E}, \mathcal{S})$ can be determined from the integral (3.5) for $m = 0$

$$N_0 = \frac{\sqrt{\lambda}}{\pi} \int_{\sqrt{R}}^{\infty} \frac{dx}{1 + x^2} \sqrt{\mathcal{E}^2 - \frac{4\mathcal{S}^2(1 + x^2)}{x^2}} \,, \qquad R^2 = \frac{4\mathcal{S}^2}{\mathcal{E}^2 - 4\mathcal{S}^2} \,. \qquad \text{(D.13)}$$

We find

$$N_0 = \frac{2\mathcal{S}\sqrt{\lambda}}{\pi} \int_1^{\infty} \frac{dx}{1 + R^2 x^2} \sqrt{1 - \frac{1}{x^2}} = \frac{1}{2}\sqrt{\lambda}(\mathcal{E} - 2\mathcal{S}) \,. \qquad \text{(D.14)}$$

Note that in the limit of $m \to 0$ (or zero tension) the solution is the same as for a point particle[10] moving along a massless geodesic in AdS_5 from some finite ρ to $\rho = \infty$. We may thus interpret the string solution with large energy (implying large spin $\mathcal{S}$ or large oscillation number $\mathcal{N}$) as being "fast" (cf. [9]). For $\mathcal{E} \sim 1$ which we considered in section 3, the string is not fast: in this case the $m \to 0$ limit corresponds to the radial coordinate ρ changing between two finite values.

The resulting expression for the oscillation number is

$$N = \frac{E}{2} - S + a_1 \lambda^{1/4} \sqrt{E} + a_2 \lambda^{3/4} \frac{1}{\sqrt{E}} + O(E^{-3/2}) \,, \qquad \text{(D.15)}$$

where

$$a_1 = \frac{\sqrt{2}k_1}{3\pi} m^{3/2} = -0.127 m^{3/2} \,, \qquad a_2 = \frac{k_2}{10\sqrt{2}\pi} m^{5/2} = 0.042 m^{5/2} \qquad \text{(D.16)}$$

Expressing energy in terms of N and S then gives (3.10). Since we expanded in large energy while keeping $\mathcal{S}$ fixed, this expression is valid for large $\mathcal{E}$ and $\mathcal{N}$, but not large $\mathcal{S}$.[11]

Let us now find the energy for large $\mathcal{S}$ and small $\mathcal{N}$. Let us start with (D.8) and expand now at large $\mathcal{S}$ and large $\mathcal{E}$, setting $\mathcal{S} = u\mathcal{E}$, with u fixed. For the solution to exist we need to satisfy (3.7) for large $\mathcal{S}$, which implies $u \le \frac{1}{2}$. Expanding in large energy $\mathcal{E}$ but with u fixed we obtain

$$N = \frac{E}{2} - S + a_1 \lambda^{1/4} \sqrt{E} \left(1 - 4\frac{S^2}{E^2}\right)^{1/4} + a_2 \lambda^{3/4} \frac{1}{\sqrt{E}} \frac{1 - 8\frac{S^2}{E^2}}{\left(1 - 4\frac{S^2}{E^2}\right)^{5/4}} + O(E^{-3/2}) \quad \text{(D.17)}$$

This reduces to (D.15) if we expand in small u. In the limit $u \to \frac{1}{2}$ we get

$$N = \frac{E}{2} - S + a_1 \lambda^{1/4} \sqrt{E} \left(\frac{E}{S} - 2\right)^{1/4} - a_2 \lambda^{3/4} \frac{1}{\sqrt{E}} \left(\frac{E}{S} - 2\right)^{-5/4} + O(E^{-3/2}) \,, \quad \text{(D.18)}$$

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[10]Note that the solution (3.1) depends on σ only in combination with m.

[11]Note that in the limit $S \to 0$ we obtain from (3.10) a different expression for the energy then found in the absence of rotation in (C.11), in spite of the fact that setting $\mathcal{S} = 0$ in (3.3) gives the same equation as in the case of string pulsating in one plane. This discontinuity is explained by the fact that taking the limit $\mathcal{S} \ne 0$ changes the behavior of the potential at $\rho = 0$. With nonzero $\mathcal{S}$ the potential blows up at $\rho = 0$, while with $\mathcal{S} = 0$ it is finite at $\rho = 0$.

i.e.

$$E = 2S + b_0 N + b_1 \lambda^{1/3} S^{1/3} + b_2 \lambda^{3/2} S^{-1/3} + ... \tag{D.19}$$

where for $m = 1$ we have $b_0 = 1.835$, $b_1 = 0.454$. For $N \to 0$, this is different from the expression (2.19) in the case of the rigid rotating string, which for $m = 1$ gives $E = 2S + 1.19(\lambda S)^{1/3} + ...$ While the S dependence is the same the coefficients are different. This is an indication of the same discontinuity we mentioned above in the case of large energy and large N and the limit $S \to 0$. In this sense there is a symmetry between the $\mathcal{N}$ and $\mathcal{S}$ dependence of the energy.

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